Geometric phase gate on an optical transition for ion trap quantum computation

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We propose a geometric phase gate of two ion qubits that are encoded in two levels linked by an optical dipole-forbidden transition. Compared to hyperfine geometric phase gates mediated by electric dipole transitions, the gate has many interesting properties, such as very low spontaneous emission rates, applicability to magnetic field insensitive states, and use of a co-propagating laser beam geometry. We estimate that current technology allows for infidelities of around $10^{-4}$. One of the important and most difficult experimental efforts of quantum computation is the realization of almost perfect two-qubit gate operations. Currently it is believed that gate error probabilities of about $10^{-3}$ would be sufficiently low to allow for so-called efficient fault-tolerant quantum computing [1,2]. Strings of trapped ions are among the most promising candidates for the realization of a quantum computer. The lowest gate infidelity experimentally achieved with ion quantum gates is still around 3% [3]. The main limitations of this geometric phase gate come from spontaneous emission and magnetic field fluctuations [3,4].

Ion trap quantum computation can be implemented with two alternative qubit encodings: hyperfine ground state qubits and qubit states connected by optical transitions. For hyperfine qubits, the gate operations are performed by Raman coupling mediated by dipole transitions. Reference [3] used an encoding based on such a hyperfine transition. In such a setting, however, it is demanding to reduce spontaneous scattering below the required fault-tolerant level [5,6], because a tremendous amount of laser power is required. Recently, the use of Raman processes on quadrupole transitions was proposed for hyperfine qubits [7]. However, this strategy requires high laser powers to achieve short gate times. Here, we propose a $\sigma^z$-type geometric phase gate on an optical transition to overcome some of the limitations present in the realization of [3]. For instance, the use of an optical quadrupole transition allows one to sufficiently reduce the likelihood of a spontaneous emission event. Also it is shown that magnetic field insensitive states can be used for the $\sigma^z$ geometric gate on an optical transition. More interestingly, the gate can be executed with a co-propagating laser beam configuration, which reduces the errors from phase fluctuations between two laser beams [4]. With hyperfine qubits, on the contrary, only a counter-propagating scheme can be utilized for the gate. Finally, the proposed gate can be directly used as a logical two-qubit gate in dephasing-free subspaces because of the natural suitability of phase gates for such purposes [8].

The gate proposed here applies to any ion-qubit states connected by weak transitions, such as quadrupole transitions of Ca$^+$, Sr$^+$, and Ba$^+$. We first show that it is possible to realize a state-dependent displacement on the optical transitions with bichromatic laser radiation, and that for the most interesting detunings of the laser fields the coupling is maximized in a co-propagating geometry. In turn, the applicability of the gate to magnetic field insensitive states is explained. We then extend the scheme to two ions and carefully study the intrinsic complications leading to infidelities of the gate. We show that these can be compensated by spin-echo techniques, thus reducing the infidelities to a level of about $10^{-4}$. We also discuss the connection between gate speed and the probability of spontaneous emissions during one gate operation, and show that the error due to it can also be reduced to a level of $10^{-4}$. Finally, we briefly examine other more technically relevant errors arising from fluctuations of experimental parameters.

Consider a single two-level ion in a one-dimensional harmonic trap interacting with two laser beams detuned from the quadrupole transition of S to D. The Hamiltonian in the interaction picture and after the rotating wave approximation with respect to optical frequencies is given by

$$\hat{H}_I = \sum_{j=1,2} \alpha_j^* \left( \frac{\hbar \Omega_j}{2} e^{-i(\Delta_j + \delta_j)} e^{i(\eta_j + \eta_j') \hat{a}^\dagger \hat{a}} + \text{H.c.} \right),$$

where $\phi^a = |D\rangle \langle S|$, $\alpha_j$, and $\alpha_j'$ are the ladder operators of the oscillator, $\nu$ is the trap frequency, and $\Omega_j$ and $\eta_j$ are the Rabi frequency and Lamb-Dicke parameter of the laser with detuning $\Delta_j$ and optical phase $\phi_j$, respectively. As will be detailed below, a state-dependent displacement operation appears by setting $\Delta_1 = \Delta$, $\Delta_2 = \Delta - \nu + \delta$ ($\delta \ll \nu$). In the Lamb-Dicke regime, at low laser intensity $\Omega_j \ll \Delta_j$, and ignoring terms faster than $\delta$, a second-order perturbation yields the following effective interaction Hamiltonian:

$$\hat{H}_{\text{eff},1} = \left( \frac{\hbar \Omega_{\text{eff}}}{2} e^{-i\delta/2} + \frac{\hbar \Omega_{\text{eff}}^*}{2} e^{i\delta/2} \right) \hat{a}^\dagger \hat{a} + \text{H.c.} \left( \frac{\hbar \Omega_{\text{eff}}}{2} e^{-i\delta/2} + \frac{\hbar \Omega_{\text{eff}}^*}{2} e^{i\delta/2} \right) \hat{a} \hat{a}^\dagger \left( \delta \ll \nu \right).$$

Here, $\Omega_{\text{eff}} = (\eta_j - \eta_j') \Delta_j / (\Delta_j + \delta_j)$, and LS denotes the light shifts coming from the transitions of carrier and the first motional sidebands and is given by $\Sigma_{j=1,2} \alpha_j \hat{a}^\dagger \hat{a}^\dagger + \alpha_j^* \hat{a} \hat{a}$. Neglecting LS for the moment, the effective Hamiltonian (2) describes the

DOI: 10.1103/PhysRevA.77.050303

PACS number(s): 03.67.Lx, 03.67.Pp, 32.80.Qk

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desired state-dependent displacement operation [4]. The time
evolution operator is found to be
\[
\hat{U}(t) = \exp \left\{ -\frac{i}{\hbar} \left[ \int_0^t dt' \hat{H}_{\text{eff}}(t') \right] \right\}
\]
from the Magnus expansion related to the Baker-Campbell-Hausdorff formula \( e^{A} e^{B} = \exp(A + B + \frac{1}{2}[A, B] + \cdots) \). With Eq. (3), the time evolution operator of Hamiltonian (2) can be obtained by \( \hat{U}_1(t) = e^{(\alpha(t)\hat{a}^\dagger\hat{a} - \alpha^* t)\hat{a}^\dagger\hat{a}} e^{(\beta(t)\hat{a}^2)} \). Here, \( \alpha(t) = \frac{n_{\text{res}}}{2\delta}(1 - e^{-\delta t}) \) and \( \Phi(t) = \frac{n_{\text{res}}}{2\delta^2} (\delta t - \sin(\delta t)) \). The ion moves periodically along a circular path in phase space of radius \( \hbar\Omega_{\text{eff}}/2\delta \) with periodicity \( 2\pi/\delta \), and the direction of motion is determined by the qubit state and \( \phi_L \). At \( t = 2\pi/\delta \), it returns to the original motional state and acquires the geometrical phase \( \Phi_{\text{g}} = 2\pi[\Omega_{\text{eff}}/2\delta^2] \). The latter phase depends only on the area enclosed by the trajectory, so both qubit states gain the same geometrical phase independently of \( \phi_L \).

We now show that, for optical-transition qubits, a co-propagating geometry maximizes the strength of the Raman coupling \( \Omega_{\text{eff}} \). For the case of hyperfine ground state qubits connected by dipole transitions, the detunings \( \Delta_1, \Delta_2 \) must be much larger than \( \nu \) in Fig. 1(a), so \( \Delta_1 = \Delta_2 = \Delta \), which implies that \( \Omega_{\text{eff}} = \frac{n_{\text{res}}}{2\Delta} \). It is then essential, in order to achieve a nonvanishing coupling, to use a nonco-propagating laser beam configuration (\( \eta_1 \neq \eta_2 \)). With quadrupole transitions, the detunings can be of the order of \( \nu \) without considerable spontaneous emission. At the detunings \( \Delta_1 = -\Delta_2 = \frac{\pi}{4} \), the coupling strength \( \Omega_{\text{eff}} = \frac{n_{\text{res}}}{\Delta} \) is maximized at \( \eta_1 = \eta_2 \). The Raman coupling with those detunings are depicted in Fig. 1(b). Most interestingly, the co-propagating geometry reduces optical phase fluctuations from path instabilities. Furthermore the co-propagation geometry also ensures that the displacement operation can be executed regardless of the ions’ spacing. This is in contrast to a counterpropagating geometry where it is necessary to carefully control such spacings so as to have the proper laser phase on each ion [3].

Moreover, the symmetry of the detunings guarantees that the light shift in Eq. (2) disappears provided that both lasers’ intensities coincide \( \Omega_1 = \Omega_2 \). Thus, it is not necessary to consider polarization states to equalize ac Stark shifts of internal states from the two laser beams. Finally, state-dependent coupling is achieved without any restriction on the magnetic field properties of the states. The scheme proposed here is, therefore, applicable to magnetic field insensitive transitions, e.g., the quadrupole transitions of \(^{43}\text{Ca}^+ \) ion [9].

Now we extend the above consideration to two ions and study the two-qubit gate operation with the detunings \( \Delta_1 = \frac{\pi}{4} - \frac{\pi}{2}, \Delta_2 = \frac{\pi}{4} + \frac{\pi}{2} \) and the same Rabi frequency \( \Omega_1 = \Omega_2 = \Omega \). We focus on the center of mass mode (CM). With two ions, the effective Hamiltonian is given by
\[
\tilde{\hat{H}}_{\text{eff},2} = \frac{\hbar}{2} \left( \hat{a} e^{i(\Delta_1 + \theta)} + \hat{b} e^{i(\Delta_2 + \theta)} \right) \hat{S}^z + \frac{4 \hbar \eta}{3} \left( \hat{a}^2 \hat{b}^2 \otimes \hat{a}^2 \hat{b}^2 \right) + \hat{O}(\eta^2),
\]
where \( \hat{S}^z = \hat{a}^2 \hat{b}^2 \). The time evolution operator is found to be
\[
\hat{U}(t) = \exp \left\{ -\frac{i}{\hbar} \left[ \int_0^t dt' \hat{H}_{\text{eff}}(t') \right] \right\}
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\tilde{\hat{H}}_{\text{eff},2} = \frac{\hbar}{2} \left( \hat{a} e^{i(\Delta_1 + \theta)} + \hat{b} e^{i(\Delta_2 + \theta)} \right) \hat{S}^z + \frac{4 \hbar \eta}{3} \left( \hat{a}^2 \hat{b}^2 \otimes \hat{a}^2 \hat{b}^2 \right) + \hat{O}(\eta^2),
\]
Therefore, one needs to increase the integration is executed with spin-echo sequence and shaping, which is 100 times smaller than the gate time. The Lamb-Dicke parameter \( \eta \) for the same laser intensity, because in all of these cases the coupling strength is that of the first sideband interaction. The maximum intensity is, however, limited by the off-resonant excitations, \( \Omega < |\Delta_{1(2)}| = \nu/2 \). The gate time \( 2\pi/\delta = 2\pi/|\Delta_{1(2)}| \), therefore, is limited by \( 2\pi/\eta\nu \). To guarantee that \( P_{sp} < 10^{-4} \), for Ca\(^+\) ions the operation time has to be faster than 100 \( \mu s \). This means that it is necessary to increase \( \Omega \) close to the detunings \( |\Delta_{1(2)}| \) as shown at dashed lines in Fig. 2(b) and also to use large \( \nu/\eta \) as shown in Fig. 2(a).

When \( \Omega \) is comparable to the detunings, the direct coupling term from off-resonant excitations, \( \hat{H}_d \) is,\( \hat{H}_d = \sum_{j=1,2} (\hat{S}_x^j + \hat{S}_y^j) (\Delta_0 e^{-i\Delta_j t} + H.c.) \), neglected in Eqs. (2) and (4), needs to be considered. The term mainly induces population exchanges between electronic states. The population error from off-resonant excitations can be described by \( 1 - (\Delta_0 \nu/4)^2 \sin^2(\pi/4) \) around \( t = 2\pi/\delta \) [10]. The infidelity can be reduced by either precise control of system parameters,
such as $\delta = \nu/2N$ with $N$ as an integer, or by pulse shaping. By using pulse shaping, one can start and end the gate operation with a fairly small $\Omega$ by adiabatically changing laser intensities [14,15]. As we can see in Fig. 2(b), the infidelity from the direct coupling can be in the order of $10^{-4}$ up to $\nu/4$ of $\Omega$ by using rise and fall times of 10 $\mu$s for the pulses.

In the simulation, for a large $\Omega$ we observed a reduction in the Raman coupling strength $\Omega_{\text{eff}}$ proportional to $(\Omega/\nu)^2$. We believe that this reduction is due to an admixture of the other electronic state due to, for instance, off-resonant excitations. The Raman coupling $\Omega_{\text{eff,S}}$ of the $|S\rangle$ state has an opposite sign as compared to the coupling $\Omega_{\text{eff,D}}$ of the $|D\rangle$ state, $\Omega_{\text{eff,S}} = -\Omega_{\text{eff,D}}$. Thus the contributions of other electronic states due to off-resonant excitations reduce the strength of the coupling $\Omega_{\text{eff}}$ proportional to $(\Omega/\nu)^2$.

Furthermore, the amount of the reduction at $\Omega_{\text{eff,S}}$ is slightly dependent on $\eta$, which might come from the Debye factor [10]. We note that the infidelities of Fig. 2(a) and 2(b) are obtained after correcting the reduction in $\Omega_{\text{eff}}$. In experiments, however, the change in $\Omega_{\text{eff}}$ due to direct coupling is not a problem at all, since the intensities of the bichromatic lasers will be determined experimentally.

We also carefully studied other experimental imperfections, such as the intensity fluctuations of both laser beams, positioning errors of the laser beams on the ions, fluctuations of the laser and the trap frequency, and the occupations of the bus mode and spectators’ modes. The proposed gate is quite robust to those imperfections similar to the geometric hyperfine gate of [3]. According to our simulations, for intensity fluctuations of about $10^{-2}$, a few tenths of kHz of laser frequency fluctuations, and a few Hz trap frequency fluctuations—less than 0.5 motional quanta of the all motional modes—allow for an infidelity on a level of $10^{-4}$.

In conclusion, we propose a $\sigma^z$ geometric phase gate for optical-transition qubits. The gate has a small spontaneous emission during the operation, and can be applied to magnetic field insensitive states. We analyze and simulate the gate in detail and show that the gate allows one to achieve a high fidelity implementation. The proposed $\sigma^z$ gates are interesting not only due to the high fidelity, but also to their applicability to decoherence-free subspace constituted by the logical qubits $|DS\rangle, |SD\rangle$ [8,16,17]. If we apply the laser beams described in the paper to two physical qubits that belong to the different logical qubits, the scheme works as the entangling gate for the two logical qubits.

We acknowledge support by the Austrian Science Fund (FWF), the European Commission (QGATES, SCALA, and CONQUEST networks), and the Institut für Quanteninformation GmbH. K.K. acknowledges funding by the Lise-Meitner program of the FWF. L.A. acknowledges the support from CAPES, FAPERJ, and the Brazilian Millennium Institute for Quantum Information.